

# Exponential fall-off Behavior of Regge Scatterings in Compactified Open String Theory

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## Abstract

We calculate massive string scattering amplitudes of compactified open string in the Regge regime. We extract the *complete* infinite ratios among high-energy amplitudes of different string states in the fixed angle regime from these Regge string scattering amplitudes. The complete ratios calculated by this indirect method include and extend the subset of ratios calculated previously [19, 20] by the more difficult direct fixed angle calculation. In this calculation of *compactified* open string scattering, we discover a realization of arbitrary real values  $L$  in the identity Eq.(4.18), rather than integer value only in all previous high-energy string scattering amplitude calculations. The identity in Eq.(4.18) was explicitly proved recently in [36] to link fixed angle and Regge string scattering amplitudes. In addition, we discover a kinematic regime with stringy highly winding modes, which shows the unusual exponential fall-off behavior in the Regge string scattering. This is in complementary with a kinematic regime discovered previously [20], which shows the unusual power-law behavior in the high-energy fixed angle compactified string scatterings.

Key words: Regge string scatterings; High-energy String

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## I. INTRODUCTION

There are three fundamental characteristics of high-energy fixed angle string scattering amplitudes [1–3], which are not shared by the field theory scattering. These are the softer exponential fall-off behavior (in contrast to the hard power-law behavior of field theory scatterings), the infinite Regge-pole structure of the form factor and the existence of infinite number of linear relations [4–13], or stringy symmetries, discovered recently among high-energy string scattering amplitudes of different string states. An important new ingredient to derive these linear relations is the zero-norm states (ZNS) [14–16] in the old covariant first quantized (OCFQ) string spectrum, in particular, the identification of inter-particle symmetries induced by the inter-particle ZNS [14] in the spectrum. Other approaches related to this development can be found in [17].

Recently, following an old suggestion of Mende [18], two of the present authors [19] calculate high-energy fixed angle massive scattering amplitudes of closed bosonic string with some coordinates compactified on the torus. The calculation was extended to the compactified open string scatterings [20]. An infinite number of linear relations among high-energy scattering amplitudes of different string states were obtained in the fixed angle or Gross kinematic regime (GR). The UV behavior in the GR shows the usual soft exponential fall-off behavior. These results are reminiscent of the existence of an infinite number of massive ZNS in the compactified closed [21] and open [22] string spectrums constructed previously. In addition, it was discovered that, for some kinematic regime with super-highly winding modes at *fixed angle*, the so-called Mende kinematic regime (MR), these infinite linear relations break down and, simultaneously, the string amplitudes enhance to hard power-law behavior at high energies instead of the usual soft exponential fall-off behavior.

In this paper, we calculate high-energy *small angle* or Regge string scattering amplitudes [23–30] of open bosonic string with one coordinates compactified on the torus. The results can be generalized to more compactified coordinates. It is shown that there is no linear relations among Regge scattering amplitudes as expected. However, as in the case of noncompactified Regge string scattering amplitude calculation [31–33], we can deduce the infinite GR ratios in the fixed angle from these compactified Regge string scattering amplitudes. We stress that the GR ratios calculated in the present paper by this indirect method from the Regge calculation are for the most general high-energy vertex rather than

only a subset of GR ratios obtained directly from the fixed angle calculation [19, 20]. In this calculation, we have used a set of master identities Eq.(4.18) to extract the GR ratios from Regge scattering amplitudes. Mathematically, the complete proof of these identities for *arbitrary real values*  $L$  was recently worked out in [36] by using an identity of signless Stirling number of the first kind in combinatorial theory. The proof of the identity for  $L = 0, 1$ , was previously given in [31–33] based on a set of identities of signed Stirling number of the first kind [35]. It is interesting to see that, physically, the identities for arbitrary real values  $L$  can only be realized in high-energy compactified string scatterings considered in this paper. All other high-energy string scatterings calculated previously [31–33] correspond to integer values of  $L$  only. A recent work on string D-particle scatterings [34] also gave integer values  $L$ .

More importantly, we discover an exponential fall-off behavior of high-energy compactified open string scatterings in a kinematic regime with highly winding modes at small angle. The existence of this regime was conjectured in [20]. However, no Regge scatterings were calculated there and thus the results for the small angle scatterings extracted from the fixed angle calculation were not completed and fully reliable [31, 32]. The discovery of the soft exponential fall-off behavior in this kinematic regime with small angle in compactified string scatterings is in complementary with a kinematic regime discovered previously [19, 20], which shows the unusual power-law behavior in the high-energy fixed angle compactified string scatterings. This paper is organized as following. In section II, we set up the kinematics. In section III, we review the fixed angle compactified string scatterings. Section IV is devoted to the compactified Regge string scatterings. We first calculate the Regge string scattering amplitudes and extract the most general fixed angle ratios from these Regge amplitudes. We then derive a Regge regime which shows an unusual exponential fall-off behavior. A brief conclusion is made in section V.

## II. KINEMATICS SET-UP

We consider 26D open bosonic string with one coordinate compactified on  $S^1$  with radius  $R$ . It is straightforward to generalize our calculation to more compactified coordinates. The

mode expansion of the compactified coordinate is

$$X^{25}(\sigma, \tau) = x^{25} + K^{25}\tau + i \sum_{k \neq 0} \frac{\alpha_k^{25}}{k} e^{-ik\tau} \cos n\sigma \quad (2.1)$$

where  $K^{25}$  is the canonical momentum in the  $X^{25}$  direction

$$K^{25} = \frac{2\pi J - \theta_l + \theta_i}{2\pi R}. \quad (2.2)$$

Note that  $J$  is the quantized momentum and we have included a nontrivial Wilson line with  $U(n)$  Chan-Paton factors,  $i, l = 1, 2 \dots n$ , which will be important in the later discussion. The mass spectrum of the theory is

$$M^2 = (K^{25})^2 + 2(N - 1) \equiv \left( \frac{2\pi J - \theta_l + \theta_i}{2\pi R} \right)^2 + \hat{M}^2 \quad (2.3)$$

where we have defined level mass as  $\hat{M}^2 = 2(N - 1)$  and  $N = \sum_{k \neq 0} \alpha_{-k}^{25} \alpha_k^{25} + \alpha_{-k}^\mu \alpha_k^\mu$ ,  $\mu = 0, 1, 2 \dots 24$ . We are going to consider 4-point correlation function in this paper. In the center of momentum frame, the kinematic can be set up to be [19, 20]

$$k_1 = \left( +\sqrt{p^2 + M_1^2}, -p, 0, -K_1^{25} \right), \quad (2.4)$$

$$k_2 = \left( +\sqrt{p^2 + M_2^2}, +p, 0, +K_2^{25} \right), \quad (2.5)$$

$$k_3 = \left( -\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi, -K_3^{25} \right), \quad (2.6)$$

$$k_4 = \left( -\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi, +K_4^{25} \right) \quad (2.7)$$

where  $p$  is the incoming momentum,  $q$  is the outgoing momentum and  $\phi$  is the center of momentum scattering angle. In the high-energy limit, one includes only momenta on the scattering plane, and we have included the fourth component for the compactified direction as the internal momentum. The conservation of the fourth component of the momenta implies

$$\sum_m K_m^{25} = \sum_m \left( \frac{2\pi J_m - \theta_{l,m} + \theta_{i,m}}{2\pi R} \right) = 0. \quad (2.8)$$

Note that

$$k_i^2 = (K_i^{25})^2 - M_i^2 = -\hat{M}_i^2. \quad (2.9)$$

We have

$$\begin{aligned} -k_1 \cdot k_2 &= \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 + K_1^{25} K_2^{25} \\ &= \frac{1}{2} (s + k_1^2 + k_2^2) = \frac{1}{2} s - \frac{1}{2} (\hat{M}_1^2 + \hat{M}_2^2), \end{aligned} \quad (2.10)$$

$$\begin{aligned} -k_2 \cdot k_3 &= -\sqrt{p^2 + M_2^2} \cdot \sqrt{q^2 + M_3^2} + pq \cos \phi + K_2^{25} K_3^{25} \\ &= \frac{1}{2} (t + k_2^2 + k_3^2) = \frac{1}{2} t - \frac{1}{2} (\hat{M}_2^2 + \hat{M}_3^2), \end{aligned} \quad (2.11)$$

$$\begin{aligned} -k_1 \cdot k_3 &= -\sqrt{p^2 + M_1^2} \cdot \sqrt{q^2 + M_3^2} - pq \cos \phi - K_1^{25} K_3^{25} \\ &= \frac{1}{2} (u + k_1^2 + k_3^2) = \frac{1}{2} u - \frac{1}{2} (\hat{M}_1^2 + \hat{M}_3^2) \end{aligned} \quad (2.12)$$

where  $s, t$  and  $u$  are the Mandelstam variables with

$$s + t + u = \sum_i \hat{M}_i^2 = 2(N - 4). \quad (2.13)$$

Note that the Mandelstam variables defined above are not the usual 25-dimensional Mandelstam variables in the scattering process since we have included the internal momentum  $K_i^{25}$  in the definition of  $k_i$ . In order to define the Regge or fixed momentum transfer regime, we define the momenta

$$\hat{k}_1 = \left( +\sqrt{p^2 + \hat{M}_1^2}, -p, 0, 0 \right), \quad (2.14)$$

$$\hat{k}_2 = \left( +\sqrt{p^2 + \hat{M}_2^2}, +p, 0, 0 \right), \quad (2.15)$$

$$\hat{k}_3 = \left( -\sqrt{q^2 + \hat{M}_3^2}, -q \cos \phi, -q \sin \phi, 0 \right), \quad (2.16)$$

$$\hat{k}_4 = \left( -\sqrt{q^2 + \hat{M}_4^2}, +q \cos \phi, +q \sin \phi, 0 \right) \quad (2.17)$$

and the corresponding 25-dimensional Mandelstam variables

$$-\hat{k}_1 \cdot \hat{k}_2 = \sqrt{p^2 + \hat{M}_1^2} \cdot \sqrt{p^2 + \hat{M}_2^2} + p^2 = \frac{1}{2} (s_{25} - \hat{M}_1^2 - \hat{M}_2^2), \quad (2.18)$$

$$-\hat{k}_2 \cdot \hat{k}_3 = -\sqrt{p^2 + \hat{M}_2^2} \cdot \sqrt{q^2 + \hat{M}_3^2} + pq \cos \phi = \frac{1}{2} (t_{25} - \hat{M}_2^2 - \hat{M}_3^2), \quad (2.19)$$

$$-\hat{k}_1 \cdot \hat{k}_3 = -\sqrt{p^2 + \hat{M}_1^2} \cdot \sqrt{q^2 + \hat{M}_3^2} - pq \cos \phi = \frac{1}{2} (u_{25} - \hat{M}_1^2 - \hat{M}_3^2) \quad (2.20)$$

where

$$s_{25} + t_{25} + u_{25} = \sum_i \hat{M}_i^2 = 2(N - 4). \quad (2.21)$$

In the high-energy limit, we define the polarizations on the scattering plane to be

$$e^P = \frac{1}{M_2} \left( \sqrt{p^2 + M_2^2}, p, 0, 0 \right), \quad (2.22)$$

$$e^L = \frac{1}{M_2} \left( p, \sqrt{p^2 + M_2^2}, 0, 0 \right), \quad (2.23)$$

$$e^T = (0, 0, 1, 0) \quad (2.24)$$

where the fourth component refers to the compactified direction. The center of mass energy  $E$  is defined as (for large  $p, q$ )

$$E = \frac{1}{2} \left( \sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) = \frac{1}{2} \left( \sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2} \right). \quad (2.25)$$

The projections of the momenta on the scattering plane can be calculated to be (here we only list the ones we will need for our calculation)

$$e^P \cdot k_1 = -\frac{1}{M_2} \left( \sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right) \quad (2.26)$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left( \sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) \quad (2.27)$$

$$e^T \cdot k_1 = 0 \quad (2.28)$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left( \sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi \right), \quad (2.29)$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left( p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \phi \right), \quad (2.30)$$

$$e^T \cdot k_3 = -q \sin \phi. \quad (2.31)$$

### III. FIXED ANGLE REGIME

We begin with a brief review of high energy string scatterings for the noncompactified 26D open bosonic string in the GR. That is in the kinematic regime  $s, -t \rightarrow \infty$ ,  $t/s \approx -\sin^2 \frac{\theta}{2} =$  fixed (but  $\theta \neq 0$ ) where  $s, t$  and  $u$  are the Mandelstam variables for the noncompactified

momenta and  $\theta$  is the 26D CM scattering angle. It was shown [7, 8] that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level  $M_2^2 = 2(N - 1)$  are of the form

$$|N, 2m, r\rangle \equiv (\alpha_{-1}^T)^{N-2m-2r} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^r |0, k_2\rangle. \quad (3.1)$$

It can be shown that the high-energy vertex in Eq.(3.1) are conformal invariants up to a subleading term in the high-energy expansion. Note that  $e^P$  approaches to  $e^L$  in the GR, and the scattering plane is defined by the spatial components of  $e^L$  and  $e^T$ . Polarizations perpendicular to the scattering plane are ignored because they are kinematically suppressed for four point scatterings in the high-energy limit. One can then use the saddle-point method to calculate the high energy scattering amplitudes. For simplicity, we choose  $k_1$ ,  $k_3$  and  $k_4$  to be tachyons and the final result of the ratios of high energy, fixed angle string scattering amplitude are [7, 8]

$$\frac{T^{(N, 2m, r)}}{T^{(N, 0, 0)}} = \left(-\frac{1}{M_2}\right)^{2m+r} \left(\frac{1}{2}\right)^{m+r} (2m-1)!! \quad (3.2)$$

We now review the results obtained previously for the compactified open string scatterings at fixed angle  $\phi = \text{finite}$  [20]. For simplicity, the second vertex was chosen to be

$$|N, 0, r, i, l\rangle = (\alpha_{-1}^T)^{N-2r} (\alpha_{-2}^L)^r |k_2, l_2, i, l\rangle \quad (3.3)$$

at mass level  $\hat{M}_2^2 = 2(N - 1)$ , which was scattered with three "tachyon" states (with  $\hat{M}_1^2 = \hat{M}_3^2 = \hat{M}_4^2 = -2$ ). The high-energy fixed angle open string scattering amplitudes with one compactified coordinate were calculated to be (the trace factor due to Chan-Paton was ignored) [20]

$$\begin{aligned} T^{(N, 0, r, i, l)} &\simeq (-iq \sin \phi)^N \left( -\frac{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)}{M_2 q^2 \sin^2 \phi} \right)^r \\ &\cdot \sum_{j=0}^r \binom{r}{j} \left[ -\frac{p(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2})}{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)} \right]^j \\ &\cdot B\left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t\right) \left(-1 - \frac{1}{2}s\right)_{N-2j} \left(-1 - \frac{1}{2}t\right)_{2j} \left(2 + \frac{1}{2}u\right)_N^{-1} \end{aligned} \quad (3.4)$$

where  $(a)_j = a(a+1)(a+2)\dots(a+j-1)$  is the Pochhammer symbol, and  $(a)_j = a^j$  for large  $a$  and fixed  $j$ .

### A. Fixed Winding Modes

In the Gross regime,  $p^2 \gg K_i^2$  and  $p^2 \gg N$ , Eq.(3.4) reduces to

$$T^{(N,0,r,i,l)} \simeq \left( -iE \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} \right)^N \left( -\frac{1}{2M_2} \right)^r \cdot B \left( -1 - \frac{1}{2}s, -1 - \frac{1}{2}t \right). \quad (3.5)$$

For each fixed mass level  $N$ , we have the linear relation for the scattering amplitudes

$$\frac{T^{(N,0,r,i,l)}}{T^{(N,0,0,i,l)}} = \left( -\frac{1}{2M_2} \right)^r \quad (3.6)$$

with ratios consistent with our previous result in Eq.(3.2). Note that in Eq.(3.5) there is an exponential fall-off factor in the high-energy expansion of the beta function. The infinite linear relation in Eq.(3.6) "soften" the high-energy behavior of string scatterings in the GR.

### B. Super-highly Winding Modes

We next consider a more interesting regime, the Mende kinematic regime (MR) [20]. For the case of  $\phi = \text{finite}$ , the only choice to achieve UV power-law behavior is to require (we choose  $(K_1^{25})^2 \simeq (K_2^{25})^2 \simeq (K_3^{25})^2 \simeq (K_4^{25})^2$ )

$$(K_i^{25})^2 \gg p^2 \gg N. \quad (3.7)$$

In order to explicitly show that this choice of kinematic regime does lead to UV power-law behavior, it was shown that in this regime

$$s = \text{constant} \quad (3.8)$$

in the open string scattering amplitudes. This in turn gives the desire power-law behavior of high-energy compactified open string scattering in Eq.(3.4). On the other hand, it can be shown that the linear relations break down as expected in this regime. For the choice of kinematic regime in Eq.(3.7), Eq.(2.10) and Eq.(3.8) imply

$$\lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{K_1^{25} K_2^{25}} = \lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{\left( \frac{2\pi l_1 - \theta_{j,1} + \theta_{i,1}}{2\pi R} \right) \left( \frac{2\pi l_2 - \theta_{j,2} + \theta_{i,2}}{2\pi R} \right)} = -1. \quad (3.9)$$

For finite momenta  $J_1$  and  $J_2$ , the power-law behavior can be achieved by scattering of string states with "super-highly" winding nontrivial Wilson lines

$$(\theta_{i,1} - \theta_{l,1}) \rightarrow \infty, \quad (\theta_{i,2} - \theta_{l,2}) \rightarrow -\infty. \quad (3.10)$$



Note that the directions of momenta  $K_1^{25}$  and  $K_2^{25}$  are *opposite*. Since  $(K_i^{25})^2 \gg p^2$  and by Eq.(2.9), we can do the expansion of Eq.(3.9) to get

$$\frac{-K_1^{25} K_2^{25} (1 + \frac{p^2}{2(K_1^{25})^2})(1 + \frac{p^2}{2(K_2^{25})^2}) + p^2}{K_1^{25} K_2^{25}} = -1, \quad (3.11)$$

which in turn, to the first order of the expansion, gives

$$-K_1^{25} K_2^{25} (1 + \frac{p^2}{2(K_1^{25})^2} + \frac{p^2}{2(K_2^{25})^2}) + p^2 = -K_1^{25} K_2^{25}. \quad (3.12)$$

A simple calculation then gives

$$(\lambda_1 + \lambda_2)^2 = 0 \quad (3.13)$$

where signs of  $\lambda_1 = \frac{p}{K_1^{25}}$  and  $\lambda_2 = -\frac{p}{K_2^{25}}$  are chosen to be the same. It can be seen now that the kinematic regime in Eq.(3.7) does solve Eq.(3.13). In conclusion, there is a  $\phi = \text{finite}$  regime with UV power-law behavior for the high-energy compactified open string scatterings. This new phenomenon never happens in the 26D string scatterings. The linear relations break down as expected in this regime.

#### IV. REGGE SCATTERINGS

We now begin to consider the compactified Regge string scatterings. It is important at this point to note that in the high-energy,  $t_{25} = \text{finite}$  approximation, all  $\hat{M}_i^2$  can be neglected and we have

$$\cos \phi \simeq 1 + \frac{t_{25}}{2s_{25}}, \quad p \sin \phi \simeq \sqrt{-t_{25}} \quad (4.1)$$

where we have used Eq.(2.18) to Eq.(2.21) to do the calculation. It is easy to see that high-energy fixed  $t_{25} = \text{finite}$  (instead of fixed  $t$ ) approximation corresponds to the *small angle*  $\phi$  or Regge regime (RR). In the high-energy limit,  $p^2 = q^2 = s_{25}/4$ . In this paper, we are going to consider two different Regge regimes (RR) corresponding to fixed winding modes  $(K_i^{25})^2 \ll p^2$  and highly winding modes  $(K_i^{25})^2 \simeq p^2$  respectively.

##### A. Fixed Winding Modes

We first consider the following RR

$$t_{25} = \text{finite}, (K_i^{25})^2 \ll p^2 \gg N. \quad (4.2)$$

In this regime

$$t_{25} \simeq t + (K_2^{25} - K_3^{25})^2. \quad (4.3)$$

A class of high-energy vertex at fixed mass level  $N = \sum_{n,m} np_n + mq_m$  are [31, 33]

$$|p_n, q_m, i, l\rangle = \prod_{l>0} (\alpha_{-n}^T)^{p_n} \prod_{m>0} (\alpha_{-m}^L)^{q_m} |k_2, l_2, i, l\rangle. \quad (4.4)$$

The conformal invariant property of the above vertex was discussed in [33]. Note that states containing operators  $(\alpha_{-n}^{25})$  are of sub-leading order in energy and are neglected. For simplicity, we will only consider the states

$$|N, 2m, r, i, l\rangle = (\alpha_{-1}^T)^{N-2m-2r} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^r |k_2, l_2, i, l\rangle. \quad (4.5)$$

at mass level  $\hat{M}_2^2 = 2(N-1)$  scattered with three "tachyon" states (with  $\hat{M}_1^2 = \hat{M}_3^2 = \hat{M}_4^2 = -2$ ). Eq.(4.5) is the most general high-energy vertex in the fixed angle regime. The vertex considered previously at fixed angle in Eq.(3.3) corresponds to  $m=0$  only and thus was not completed. The relevant kinematics can be calculated to be

$$e^P \cdot k_1 \simeq -\frac{s_{25}}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{t_{25} - M_2^2 - M_3^2}{2M_2} = -\frac{\tilde{t}_{25}}{2M_2}; \quad (4.6)$$

$$e^L \cdot k_1 \simeq -\frac{s_{25}}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{t_{25} + M_2^2 - M_3^2}{2M_2} = -\frac{\tilde{t}'_{25}}{2M_2}; \quad (4.7)$$

and

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t_{25}}. \quad (4.8)$$

We are now ready to calculate the Regge scattering amplitudes. Note that  $e^P \neq e^L$  in the RR [31–33]. We will calculate  $e^L$  amplitudes in this paper. The corresponding  $e^P$  amplitudes can be similarly calculated. The  $s-t$  channel of the compactified Regge string scattering amplitudes in the regime Eq.(4.2) can be calculated to be (We will ignore the trace factor due to Chan-Paton in the scattering amplitude calculation). This does not affect our final

results in this paper)

$$\begin{aligned}
A^{(N,2m,r,i,l)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[ \frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2r} \\
&\cdot \left[ \frac{e^L \cdot k_1}{-x} + \frac{e^L \cdot k_3}{1-x} \right]^{2m} \left[ \frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right]^r \\
&\simeq (\sqrt{-t_{25}})^{N-2m-2r} \left( \frac{\tilde{t}'_{25}}{2M_2} \right)^r \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3 - N + 2m} \\
&\cdot \sum_{j=0}^{2m} \binom{2m}{j} \left( \frac{s_{25}}{2M_2 x} \right)^j \left( \frac{-\tilde{t}'_{25}}{2M_2(1-x)} \right)^{2m-j} \\
&= (\sqrt{-t_{25}})^{N-2m-2r} \left( \frac{\tilde{t}'_{25}}{2M_2} \right)^r \left( \frac{\tilde{t}'_{25}}{2M_2} \right)^{2m} \\
&\cdot \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j \left( \frac{s_{25}}{\tilde{t}'_{25}} \right)^j B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 - N + j + 1). \quad (4.9)
\end{aligned}$$

Note that the term  $\frac{e^L \cdot k_1}{x^2}$  in the bracket is subleading in energy and can be neglected. In the high-energy limit, the beta function in Eq.(4.9) can be approximated by

$$B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 - N + j + 1) \simeq B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{-j} \left(-1 - \frac{t}{2}\right)^j. \quad (4.10)$$

Finally, the leading order amplitude in the RR can be written as

$$\begin{aligned}
A^{(N,2m,r,i,l)} &= B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sqrt{-t_{25}}^{N-2m-2r} \left(\frac{1}{2M_2}\right)^{2m+r} \\
&2^{2m} (\tilde{t}'_{25})^r U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'_{25}}{2}\right), \quad (4.11)
\end{aligned}$$

which is UV power-law behaved as  $t = \text{finite}$  in the beta function by Eq.(4.3).  $U$  in Eq.(4.11) is the Kummer function of the second kind and is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[ \frac{M(a, c, x)}{(a-c)!(c-1)!} - \frac{x^{1-c} M(a+1-c, 2-c, x)}{(a-1)!(1-c)!} \right] \quad (c \neq 2, 3, 4, \dots) \quad (4.12)$$

where  $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{(c)_j j!}$  is the Kummer function of the first kind.  $U$  and  $M$  are the two solutions of the Kummer equation

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \quad (4.13)$$

It is crucial to note that, in our case of Eq.(4.11),  $c = \frac{t}{2} + 2 - 2m$  and is not a constant as in the usual definition, so  $U$  in Eq.(4.11) is *not* a solution of the Kummer equation.

It is important to note that there is no linear relation among high-energy string scattering amplitudes of different string states for each fixed mass level in the RR as can be seen from Eq.(4.11). This is very different from the result in the GR in Eq.(3.2). In other words, the ratios  $A^{(N,2m,r,i,l)}/A^{(N,0,0,i,l)}$  are  $\tilde{t}'_{25}$ -dependent functions. In particular, we can extract the coefficients of the highest power of  $\tilde{t}'_{25}$  in  $A^{(N,2m,r,i,l)}/A^{(N,0,0,i,l)}$ . We can use the identity of the Kummer function

$$\begin{aligned}
& 2^{2m}(\tilde{t}'_{25})^{-2m} U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'_{25}}{2}\right) \\
&= {}_2F_0\left(-2m, -1 - \frac{t}{2}, -\frac{2}{\tilde{t}'_{25}}\right) \\
&\equiv \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right)_j \frac{\left(-\frac{2}{\tilde{t}'_{25}}\right)^j}{j!} \\
&= \sum_{j=0}^{2m} \binom{2m}{j} \left(-L - \frac{\tilde{t}'_{25}}{2}\right)_j \left(\frac{2}{\tilde{t}'_{25}}\right)^j
\end{aligned} \tag{4.14}$$

to get

$$\begin{aligned}
\frac{A^{(N,2m,r,i,l)}}{A^{(N,0,0,i,l)}} &= (-1)^m \left(-\frac{1}{2M_2}\right)^{2m+r} (\tilde{t}'_{25} - M_2^2 + M_3^2)^{-m-r} (\tilde{t}'_{25})^{2m+r} \\
&\cdot \sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'_{25}}{2}\right)_j \frac{(-2/\tilde{t}'_{25})^j}{j!} + O\left\{\left(\frac{1}{\tilde{t}'_{25}}\right)^{m+1}\right\}
\end{aligned} \tag{4.15}$$

where

$$L = 1 - N - (K_2^{25})^2 + K_2^{25} K_3^{25}. \tag{4.16}$$

If the leading order coefficients in Eq.(4.15) extracted from the high energy string scattering amplitudes in the RR are to be identified with the complete ratios in Eq.(3.2) calculated previously among high energy string scattering amplitudes in the GR [31, 32]

$$\lim_{\tilde{t}'_{25} \rightarrow \infty} \frac{A^{(N,2m,r,i,l)}}{A^{(N,0,0,i,l)}} = \left(-\frac{1}{M_2}\right)^{2m+r} \left(\frac{1}{2}\right)^{m+r} (2m-1)!! = \frac{T^{(N,2m,r,i,l)}}{T^{(N,0,0,i,l)}}, \tag{4.17}$$

we need the following identity

$$\begin{aligned}
& \sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'_{25}}{2}\right)_j \frac{(-2/\tilde{t}'_{25})^j}{j!} \\
&= 0(-\tilde{t}'_{25})^0 + 0(-\tilde{t}'_{25})^{-1} + \dots + 0(-\tilde{t}'_{25})^{-m+1} + \frac{(2m)!}{m!} (-\tilde{t}'_{25})^{-m} + O\left\{\left(\frac{1}{\tilde{t}'_{25}}\right)^{m+1}\right\}.
\end{aligned} \tag{4.18}$$

Note that the ratios calculated previously at fixed angle in Eq.(3.6) corresponds to  $m = 0$  only in Eq.(4.17) and thus was not completed. The ratios in Eq.(4.17) calculated in this paper by the indirect method through the RR amplitudes are the most general ones. The coefficient of the term  $O \left\{ (1/\tilde{t}'_{25})^{m+1} \right\}$  in Eq.(4.18) is irrelevant for our discussion. The proof of Eq.(4.18) turns out to be nontrivial. The standard approach by using integral representation of the Kummer function seems not applicable here. Presumably, the difficulty of the rigorous proof of Eq.(4.18) is associated with the nonconstant  $c$  mentioned previously.

Mathematically, the complete proof of Eq.(4.18) for *arbitrary real values*  $L$  was recently worked out in [36] by using an identity of signless Stirling number of the first kind in combinatorial theory. The proof of the identity for  $L = 0, 1$ , was previously given in [31–33] based on a set of identities of signed Stirling number of the first kind [35]. It is interesting to see that, physically, the identities for arbitrary real values  $L$  can only be realized in high-energy compactified string scatterings considered in this paper. This is due to the dependence of the value  $L$  on winding momenta  $K_i^{25}$ . All other high-energy string scattering amplitudes calculated previously [31–33] correspond to integer value of  $L$  only.

## B. Highly Winding Modes

In this subsection, we consider the more interesting RR

$$t_{25} = \text{finite}, (K_i^{25})^2 \simeq p^2 \gg N. \quad (4.19)$$

In this regime, Eq.(2.11) and Eq.(2.19) imply

$$t_{25} \simeq t - \sqrt{p^2 + \hat{M}_2^2} \cdot \sqrt{q^2 + \hat{M}_3^2} + \sqrt{p^2 + M_2^2} \cdot \sqrt{q^2 + M_3^2} - K_2^{25} K_3^{25}. \quad (4.20)$$

It's easy to see that in general<sup>1</sup>  $t$  is as large as  $p^2$  in this regime. The most general high-energy vertex at each fixed mass level  $N$  are

$$|N, 2m, r, i, l\rangle = (\alpha_{-1}^T)^{N-2m-2r} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^r |k_2, l_2, i, l\rangle. \quad (4.21)$$

Note that states containing operators  $(\alpha_{-n}^{25})$  are again of sub-leading order in energy. For simplicity, we will only consider the states

$$|N, 0, r, i, l\rangle = (\alpha_{-1}^T)^{N-2r} (\alpha_{-2}^L)^r |k_2, l_2, i, l\rangle \quad (4.22)$$

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<sup>1</sup> For some regime,  $t$  can be finite. For example, for  $K_2^{25} = K_3^{25} \simeq p^2 \gg N, t \simeq t_{25} = \text{finite}$  by Eq.(4.20).

at mass level  $\hat{M}_2^2 = 2(N-1)$  scattered with three "tachyon" states (with  $\hat{M}_1^2 = \hat{M}_3^2 = \hat{M}_4^2 = -2$ ). The  $s-t$  channel of the high-energy scattering amplitude can be calculated to be

$$A^{(N,0,r,i,l)} = \int d^4x \cdot \prod_{i < j} (x_i - x_j)^{k_i \cdot k_j} \cdot \left[ \frac{ie^T \cdot k_1}{x_1 - x_2} + \frac{ie^T \cdot k_3}{x_3 - x_2} + \frac{ie^T \cdot k_4}{x_4 - x_2} \right]^{N-2r} \cdot \left[ \frac{e^L \cdot k_1}{(x_1 - x_2)^2} + \frac{e^L \cdot k_3}{(x_3 - x_2)^2} + \frac{e^L \cdot k_4}{(x_4 - x_2)^2} \right]^r. \quad (4.23)$$

After fixing the  $SL(2, R)$  gauge and using the kinematic relations Eq.(2.26) to Eq.(2.31) and Eq.(4.1) derived previously, we have

$$\begin{aligned} A^{(N,0,r,i,l)} &= (-i\sqrt{-t_{25}})^N \left( \frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{M_2 t_{25}} \right)^r \\ &\cdot \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3 - N + 2r} \\ &\cdot \left[ \frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2})x^2} - \frac{1}{(1-x)^2} \right]^r \\ &= (-i\sqrt{-t_{25}})^N \left( \frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{M_2 t_{25}} \right)^r \\ &\cdot \sum_{j=0}^r \binom{r}{j} \left[ -\frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}} \right]^j \cdot \int_0^1 dx \cdot x^{k_1 \cdot k_2 - 2j} (1-x)^{k_2 \cdot k_3 - N + 2j} \\ &= (-i\sqrt{-t_{25}})^N \left( \frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{M_2 t_{25}} \right)^r \\ &\cdot \sum_{j=0}^r \binom{r}{j} \left[ -\frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}} \right]^j \\ &\cdot B\left(-\frac{1}{2}s + N - 2j - 1, -\frac{1}{2}t + 2j - 1\right) \end{aligned} \quad (4.24)$$

where  $B(u, v)$  is the Euler beta function. We can do the high-energy approximation of the gamma function  $\Gamma(x)$  and end up with

$$\begin{aligned}
A^{(N,0,r,i,l)} &= (-i\sqrt{-t_{25}})^N \left( \frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{M_2 t_{25}} \right)^r \\
&\cdot \sum_{j=0}^r \binom{r}{j} \left[ \frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}} \right]^j \\
&\cdot \frac{\Gamma(-1 - \frac{1}{2}s + N - 2j) \Gamma(-1 - \frac{1}{2}t + 2j)}{\Gamma(2 + \frac{1}{2}u)} \\
&\simeq (-i\sqrt{-t_{25}})^N \left( \frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{M_2 t_{25}} \right)^r \\
&\cdot \sum_{j=0}^r \binom{r}{j} \left[ \frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}} \right]^j \\
&\cdot B\left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t\right) \left(-1 - \frac{1}{2}t\right)_{2j} \left(-1 - \frac{1}{2}s\right)^{N-2j} \left(2 + \frac{1}{2}u\right)^{-N} \\
&= \left(i\frac{\sqrt{-t_{25}}}{(\frac{u}{s})}\right)^N \left( \frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{M_2 t_{25}} \right)^r \cdot B\left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t\right) \\
&\cdot \sum_{j=0}^r \binom{r}{j} \left[ -\frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}} \frac{4}{s^2} \right]^j \left(-1 - \frac{1}{2}t\right)_{2j} \quad (4.25)
\end{aligned}$$

Finally, since  $t$  is as large as  $p^2$  in the regime Eq.(2.26), we can easily do the summation and end up with

$$\begin{aligned}
A^{(N,0,r,i,l)} &= \left(i\frac{\sqrt{-t_{25}}}{(\frac{u}{s})}\right)^N \left(-\frac{1}{M_2}\right)^r \cdot B\left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t\right) \\
&\left[ -\frac{p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2}}{t_{25}} + \frac{p\sqrt{p^2 + M_1^2} + p\sqrt{p^2 + M_2^2}}{t_{25}} \left(\frac{t}{s}\right)^2 \right]^r \quad (4.26)
\end{aligned}$$

where  $(\frac{t}{s})$  and  $(\frac{u}{s})$  are fixed numbers. Since  $t$  is as large as  $s$  in this regime, the beta function in Eq.(4.26) implies that the UV behavior of the amplitude is exponential fall-off. On the other hand, it is clearly that there is no linear relation in this regime. In conclusion, we have discovered a  $\phi \simeq 0$  regime with UV exponential fall-off behavior for the high-energy compactified open string scatterings. This new phenomenon never happens in the 26D string scatterings.

## V. CONCLUSION

In this paper, we have mainly achieved three new results for high-energy string scattering amplitudes. First, we calculate massive string scattering amplitudes of compactified open string in the Regge regime. We can then extract the *complete* infinite ratios among high-energy amplitudes of different string states in the fixed angle regime from these Regge string scattering amplitudes. The complete ratios calculated by this indirect method include and extend the subset of ratios calculated previously [19, 20] by the more difficult direct fixed angle calculation.

Secondly, by studying the high-energy string scattering for the *compactified* open string, we discover in this paper a realization of arbitrary real values  $L$  in the identity Eq.(4.18) which was proposed recently to link fixed angle and Regge string scattering amplitudes. All other high-energy string scatterings calculated previously [31, 33, 34] correspond to integer value of  $L$  only. Physically, The parameter  $L$  is related to the mass level of an excited string state and can take non-integer values for Kaluza-Klein modes. Mathematically, the identity in Eq.(4.18) was explicitly proved recently for arbitrary real values  $L$  in [36] by using the signless Stirling number in combinatorial theory.

Finally, we discover a kinematic regime which shows the unusual exponential fall-off behavior in the small angle scattering. This is in complementary with a fixed angle regime discovered previously [20], which shows the unusual power-law behavior in the compactified string scatterings.

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